MTH 295
Fall 2019
Homework 8
Due Thursday, 11/7

Name: Key

1) Transform the equation into an equivalent system of first order equations. Your variables are $x_1, x_2, x_3, \ldots$.

$x''' = x'' + (x')^2 + \cos t$, where $x = x(t)$.

Let $x_1 = x$
$x_1' = x_2 = x'$
$x_2' = x_3 = x''$

So
$$\boxed{\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = x_3 + x_2^2 + \cos t \end{cases}}$$

2) Just as an n^{th} order equation can be transformed to a system of n first order equations, a system of m n^{th} order equations can be transformed into a system of mn first order equations. Transform the system of two second order equations into a system of first order equations. Again, your variables are $x_1, x_2, x_3, \ldots$

$x''+3x'+4x-2y=0,\ y''+2y'-3x+y=e^t\cos t$, where $x=x(t), y=y(t)$

2 answers are fine –

$$\left.\begin{array}{l} x_1 = x \\ x_1' = x_2 = x' \\ x_3 = y \\ x_4 = x_3' = y' \end{array}\right\} \Rightarrow \boxed{\begin{array}{l} x_1' = x_2 \\ x_2' = -3x_2 - 4x_1 + 2x_3 \\ x_3' = x_4 \\ x_4' = -2x_4 + 3x_1 - x_3 + e^t\cos t \end{array}}$$

or –

$$\begin{array}{l} x_1 = x \\ x_2 = y \\ x_3 = x_1' = x' \\ x_4 = x_2' = y' \end{array} \Rightarrow \boxed{\begin{array}{l} x_1' = x_3 \\ x_2' = x_4 \\ x_3' = -3x_3 - 4x_1 + 2x_2 \\ x_4' = -2x_4 + 3x_1 - x_2 + e^t\cos t \end{array}}$$

3) a) Find the general solution to $x' = -y, y' = 10x - 7y$ using the method of elimination.

elf $y = -x'$

$y' = -x''$

then $-x'' = 10x - 7(-x')$

$x'' + 7x' + 10x = 0$

$x = e^{kx} \Rightarrow$

$k^2 + 7k + 10 = 0$

$(k+2)(k+5) = 0$

$k = -2, k = -5$

so $x = c_1 e^{-2t} + c_2 e^{-5t}$

and $y = -x'$

$= 2c_1 e^{-2t} + 5c_2 e^{-5t}$

and the solution is -

$$\boxed{\begin{aligned} x &= c_1 e^{-2t} + c_2 e^{-5t} \\ y &= 2c_1 e^{-2t} + 5c_2 e^{-5t} \end{aligned}}$$

b) What happens to all solutions as $t \to \infty$? You should find that all solutions approach the same point (x, y). This is an example of what is called a "fixed point".

as $t \to \infty$, $e^{-2t} \to 0$ and $e^{-5t} \to 0$

so $(x, y) \to (0,0)$ as $t \to \infty$, i.e. $(0,0)$ is a fixed point.

c) Find the particular solution to the IVP consisting of the above equation and the conditions $x(0)=2, y(0)=-7$

from part a) -

$x = c_1 e^{-2t} + c_2 e^{-5t}$

$y = 2c_1 e^{-2t} + 5c_2 e^{-5t}$

$x(0) = c_1 + c_2 = 2$

$y(0) = 2c_1 + 5c_2 = -7$

so $-3c_2 = 11$

$c_2 = -11/3$

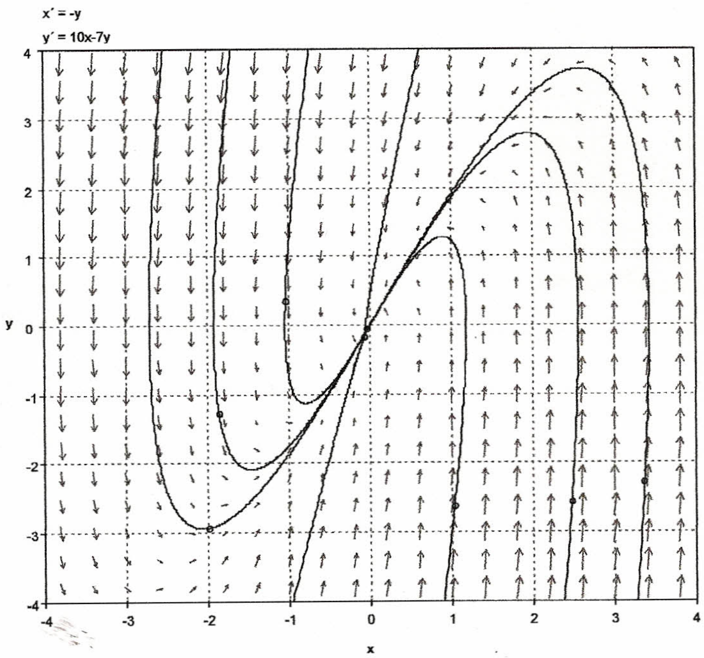
$c_1 = 2 - c_2$

$= 2 + \frac{11}{3}$

$= 17/3$

so $\boxed{x = \frac{17}{3}e^{-2t} - \frac{11}{3}e^{-5t} \qquad y = \frac{34}{3}e^{-2t} - \frac{55}{3}e^{-5t}}$

d) Include a graph of the phase plane of the above system including a few trajectories. I suggest using pplane and attaching the graph. Doing this by hand is ridiculous.



4) Matrices have strange properties. Suppose $A_1 = \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$.

a) Show that $A_1B = A_2B$. Note that $A_1B = A_2B$ does not imply that $A_1 = A_2$ or $B = 0$ as it does for algebraic variables.

$$A_1B = \begin{pmatrix} 2 & 1 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 10 \\ -4 & -8 \end{pmatrix}$$

and $A_2B = \begin{pmatrix} 1 & 3 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 10 \\ -4 & -8 \end{pmatrix}$

so $A_1B = A_2B$ even though $A_1 \neq A_2$!

b) Let $A = A_1 - A_2$ and show that $AB = 0$. Note that this implies that the product of two non-zero matrices may be equal to zero, something that can not happen with algebraic variables.

$$A = A_1 - A_2 = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$$

and $AB = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ although $A \neq 0, B \neq 0$!!

c) Show that $\det(AB) = \det(A)\det(B)$. This is a general result for any two square matrices of the same order (they must be of the same order for multiplication to be defined).

$AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ so $\det(AB) = 0 - 0 = 0$

$\det(B) = 4 - 4 = 0$

$\det(A) = 4 - 4 = 0$

so $\det(A)\det(B) = 0 \cdot 0 = 0 = \det(AB)$ //.

Boring